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Further Study on Oblique Axis Parabolic Interpolation for Curve Smoothing

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ABSTRACT: It is more than 30 years since the publication *Oblique Axis Parabolic Interpolation for Curve Smoothing* which has received approval and much attention in scientific field. Many scholars have been engaged in application of this interpolation principle sufficiently and in optimizing the process of its computation. They have concentrated themselves on researching the approximated solutions of canonical cubic algebraic equation and the refining methods to replace the original classical Cardan method of solving cubic algebraic equation. Recently, the author has found two simple and efficient mathematical methods for solving the cubic algebraic equation, i.e. the trigonometric solution based on Cardan formula and the excellent Shengjin formula created by a young Chinese mathematician called Fan Shengjin. Both of these methods have been tested with the results of sufficiently accurate solutions. Another result of this research lies in the establishing the algorithm of defining the true azimuthal angle of oblique axis parabola, which had to be solved by probing methods previously.

KEYWORDS: Oblique axis parabola, Smoothing interpolation, Cardan formula, Shengjin formula, Cubic equation

1 OSCILLATION CHARACTER OF POLYNOMIAL INTERPOLATION

The essence of interpolation is function approximation. Due to the simplicity of operation in calculating polynomials such as addition, subtraction, multiplication, derivation, we usually use a relatively simple polynomial function to approximate the original function which is more complex or not convenient for calculation. So polynomials are often used as interpolation functions.

However, the polynomial interpolation has a serious of drawbacks, e.g., the redundancy of oscillation for interpolated curves. The greater is the number of the original data points, through which the curve must pass, the higher would be the order of interpolation polynomial, which leads to the higher frequency and greater amplitude of oscillation for interpolated curves. Such oscillation is called Runge phenomenon or Runge oscillation. In addition, the higher the order of polynomial is, the more difficult is to master the physical meaning of its coefficients. Therefore, high ordered polynomial (7, 8 or higher) interpolations are rarely used in practice.

2 THE BASIC REQUIREMENTS OF CARTOGRAPHIC CURVE INTERPOLATION

Taking into account the characteristics of automatic map drawing, we can propose the following basic requirements for smoothing curves by interpolation:

(1) The smoothed curve must strictly pass through every original given vertex (interpolation point);

(2) Interpolation functions must be derivable at all nodes, that is, there is a common tangent at every intermediate node;

(3) Smoothed curves tend to go through the shortest paths, which are subjected to specific mathematical constraints;

(4) The turning points on smoothed curves (or maximal curvature points) should locate at all original given vertices.

3 PROPOSAL OF OBLIQUE AXIS PARABOLIC INTERPOLATION

3.1 Comparison between oblique axis parabola and normal axis parabola

Because a normal axis parabolic or high ordered polynomial interpolation tends to produce unwanted oscillations, we have to explore other ways. At this point, we firstly compare the oblique axis parabola with the normal one. With the different distribution of 3 non-collinear points A, B, and C, the oblique axis parabola and the normal one through these 3 points are shown in Figure 1.

In Figure 1, the oblique axis parabolic arc is the curve ABC, which goes through the three points A, B and C, with its maximal curvature at the middle point B, the relatively second point. Meanwhile, the normal axis parabola goes through the three node points is shown as ABdC in Figure 1, with its maximal curvature at point d which is not the originally given node.

3.2 Birth of the oblique axis parabolic interpolation method

Figure 1 shows that a suitable path through three ordered points A, B and C is an oblique axis parabolic curve which has the maximal curvature at point B while goes through the first and last points A and C. To generate such a parabola, we can use coordinate translation which makes the acme of the parabola locate at the middle point B, and at same times through the axis rotation, makes the parabola pass through the other points A and C. This parabola is an oblique axis parabola in a local coordinate system. The orientation angle of this axis is temporarily unknown and has to be determined. However, it can be obtained by solving the canonical cubic equation determined by the three given points. An oblique axis parabolic curve goes through the known point A, B, C can be a suitable path. Path problem is a problem irrelevant to the coordinate system, i.e., an oblique axis parabola is a path curve independent of any coordinate system.

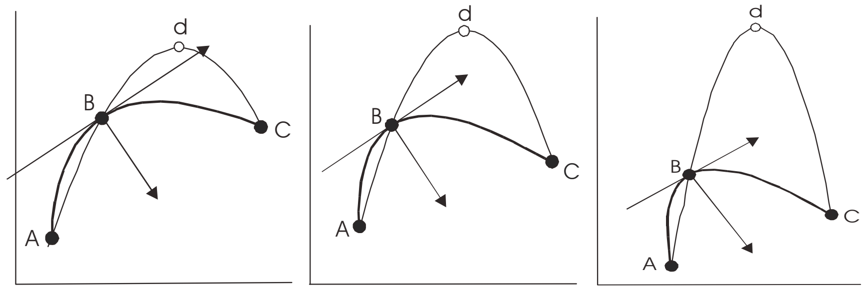

Figure 1 Comparison between oblique axis parabola and normal axis parabola

In order to draw the smooth curve better, the author proposed the oblique axis parabolic smooth interpolation 31 years ago [1]. The core problem can be revealed by the following two major issues:

3.2.1 Determination of the rational tangent direction fitting each node of the curve

Based on the theory of differential geometry, each point on a curve has its specific curvature. This property should belong to an infinitesimal of a point neighborhood on a curve. Derivative of a function has the same case. It describes the local character of a function at a specific point with its neighborhood on the corresponding curve. Here the author explains the "local" character of a derivative as follows: Suitable direction of the tangent at each curve node of discrete data is only related to the nearby nodes, and it is irrelevant to all other distant nodes. Drawing a piecewise smooth curve passing through a series of discrete points along the edge of a spline template is just based on this idea.

3.2.2 Determination of suitable interpolation function between nodes

In order to satisfy the requirements of Hermite interpolation with derivatives, the interpolation function between adjacent nodes should be subjected to concrete specific mathematical constraints on the curved path, making it tend to be the shortest. Below, except the first and last points, the whole internal interpolated arc path is located in the shuttle - shaped strip which is consisted of the second half arc of the preceding oblique axis parabola and the first half arc of the consequent oblique axis parabolic arc. However, ordinary methods which consider only the derivatives at related node points cannot ensure the interpolation results to lie within the above mentioned shuttle shaped strip.

3.2.3 Mathematical model of the oblique axis parabolic interpolation

The definition of oblique axis parabola has been mentioned above, and its mathematical model can be visually represented by Figure 2.

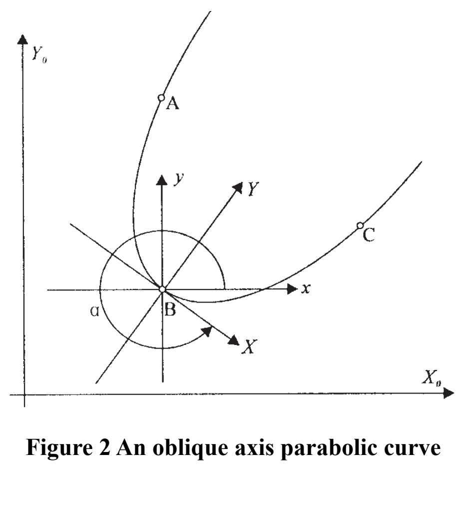

Figure 2 An oblique axis parabolic curve

The main formulae are in paper [1].

$$atg^3\alpha+btg^2\alpha+ctg\alpha+d=0 \quad (1)$$

Here,

$$\left.\begin{aligned} a&=y_C^2x_A-y_A^2x_C \\ b&=(y_C-y_A)(2x_Ax_C-y_Ay_C) \\ c&=(x_C-x_A)(x_Ax_C-2y_Ay_C) \\ d&=x_A^2y_C-x_C^2y_A \end{aligned}\right\} \quad (2)$$

In formula (2), the values x_A, y_A; x_C, y_C are the coordinates of node points A and C after translation of coordinate origin to the middle point B.

Equation (1), as a cubic algebraic equation, can be solved by the Cardan method, which provides three root values of the angle α, that is, α_1 ,α_2 and α_3. One should choose one which fit the requirement of the oblique axis parabola ($X_A<0$ and $X_C>0$, $\frac{Y_A}{X_A^2}=\frac{Y_C}{X_C^2}$) as the true angle α.

3.2.4 The main types of smooth interpolation

When the number of given vertices for a curve is more than three, through every adjacent two of all intermediate node points, except the endpoints, two different oblique axis parabola can be constructed. The main types of smooth interpolation can be generally categorized in four situations.

3.2.4.1 Curve to curve interpolation

Curve to curve interpolation is that from a curve arc to another. It is the most common one (Figure 3). The interpolation, which is from the rear half arc φ_2 of the preceding parabola to the front half arc φ_1 of the consequent parabola, is called the curve to curve interpolation. Under such circumstances, we can use a cubic arc to conjoin P_i and P_{i+1}. Here we require the cubic interpolation curve not only to approximate the known tangents, that is, the local abscissa axes of the oblique axis parabola T_iT_i and $T_{i+1}T_{i+1}$ at the both ends, but also to lie within the shuttle-shaped strip surrounded by the two adjacent parabolic arcs, in order to restrain the path of the curve. This shows that the oblique axis parabolic interpolation is a piecewise interpolation with derivatives, with certain frame tangent directions at all given nodes. Each tangent at every given node point can be determined as the abscissa axis of oblique axis parabola in local coordinate system whose origin lies on this node point.

3.2.4.2 Curve to straight interpolation

Curve to straight interpolation is the transition interpolation from the rear half φ_2 of the preceding parabola to the following straight line segment ($\varphi_1=0$) (Figure 4).

3.2.4.3 Straight to curve interpolation

Straight to curve interpolation is a transition interpolation from the straight line ($\varphi_2=0$) to the front half arc of the consequent parabola φ_1 (Figure 5).

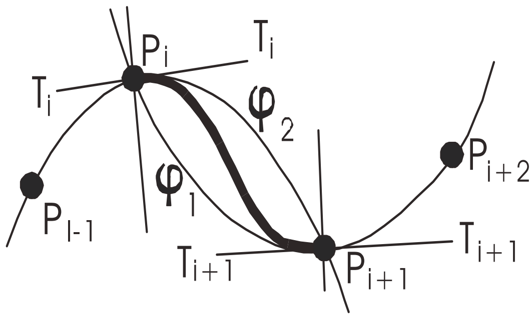

Figure 3 Formation of a serpentine curve

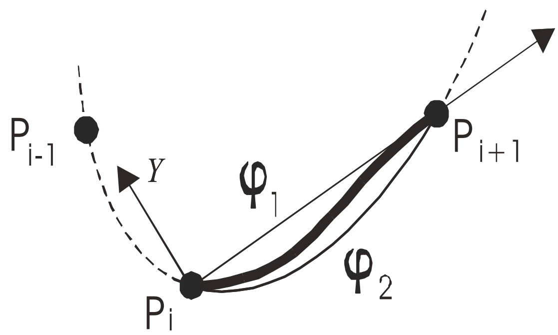

Figure 4 Curve to straight interpolation

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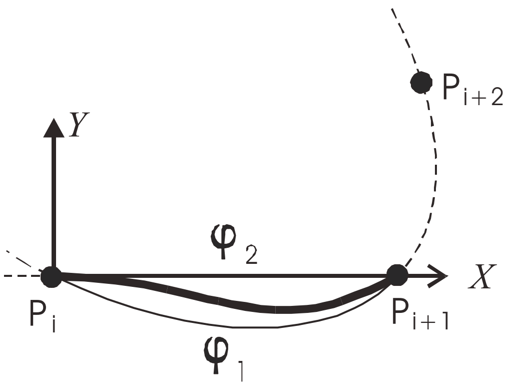


Figure 5 Straight to curve interpolation

3.2.4.4 Convex to convex connection (convexity maintaining)

In order to keep the strict convexity of successive convex curve segments, we can deal with it as follows:

When two different parabolic arcs S_i and S_{i+1} between two adjacent nodes intersect each other at point M (Figure 6(a)), find two points t_i and t_{i+1} on the arc S_i and S_{i+1}, whose tangents T_i and T_{i+1} are parallel to the chords (base lines) P_i M and MP_{i+1} respectively, then an interpolation with derivatives between t_i and t_{i+1}, composing a convex curve arc $P_i\, t_i\, t_{i+1}\, P_{i+1}$ can be made.

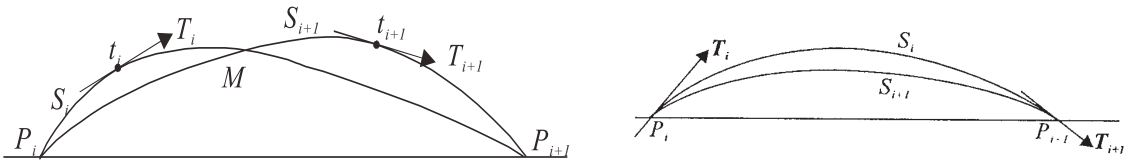
In order to achieve different degrees of convexity, slight movement from t_i and t_{i+1} towards P_i and P_{i+1} respectively is acceptable. The greater the movement is, the higher the degree of convexity of curve arc appears.

When the two parabolic arcs S_i and S_{i+1} do not intersect (Figure 6(b)), we can take the average of the two arcs S_i and S_{i+1} as the final result of interpolation.

4 ADAPTABILITY TEST WITH SEVERAL TYPICAL ORIGINAL DATA POINTS

4.1 Smooth interpolation using typical data from the reference (Junkins J.L. & Jancaitis J.R.)

In the paper [2], a line graph called "typical data" was given, requiring the software to output an "esthetically pleasing" line.



(a) Two arcs S_i and S_{i+1} intersect each other (b) Two arcs S_i and S_{i+1} do not intersect

Figure 6 Convexity maintaining

Table 1 The node coordinates of Figure 7

No.	x	y	No.	x	y
1	3.0	12.0	11	14.	5.
2	5.0	18.5	12	12.	3.
3	8.5	18.5	13	10.	2.
4	10.	18.	14	9.	2.
5	12.	17.	15	6.	4.
6	17.	14.	16	6.	5.
7	18.	14.	17	5.	9.
8	18.	10.	18	5.	13.
9	15.	7.5	19	4.	14.
10	14.	7.	1	3.0	12.0

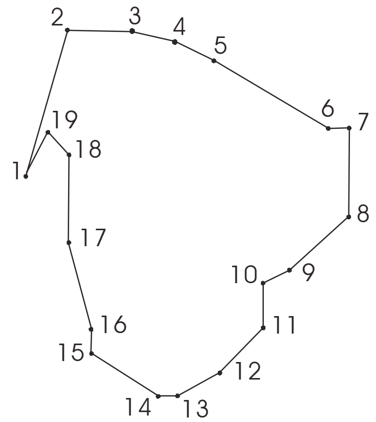
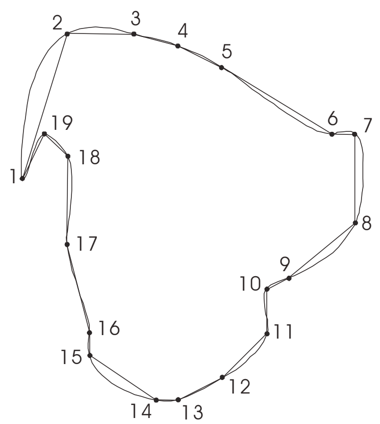


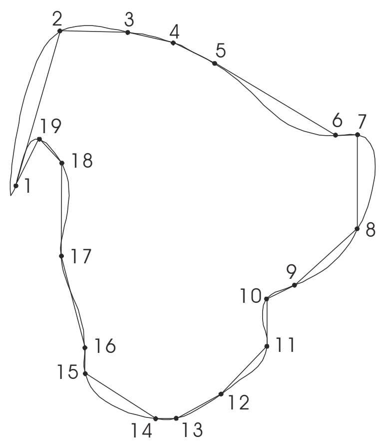
Figure 7 "Typical data" of line graph given in the paper by Junkins J.L. & Jancaitis J.R

In Figure 7, the sharp nose form coming out from the data point 19-1-2 at the upper left corner is the most rigorous test part. This sharp nose figure was initially the starting point and the primary aim to be achieved for the oblique axis parabolic interpolation method.

According to the oblique axis parabolic interpolation for curve smoothing invented by the author, an experiment using the "typical data" was carried out and the result is shown in Figure 8(a). To test the applicability of oblique axis parabola, we compare this result with the smoothed one (Figure 8(b)) from the paper [2] (nkins J.L. & Jancaitis J.R).



(a) Result of the oblique axis parabolic interpolation for curve smoothing



(b) The smoothed result in the paper by Junkins J.L. & Jancaitis J.R

Figure 8 Comparison between the smoothed results from WU Hehai and Junkins & Jancaitis

Through the comparison between two smoothed figures in Figure 8, we can find two major differences, though both figures are generally similar in appearance:

(1) In Figure 8(b), the curve turning points (points with the maximal curvature) are not located on the original data points (nodes), most apparently at point 1 and point 7.

(2) In Figure 8(b)**,** due to the first drawback, the second drawback appears, namely, the length of the curve path between two successive nodes has been improperly increased.

Figure 9 is the amplified expression to see whether the original data (nodes) locate at the turning points (points with maximum curvatures), e.g., point 1.

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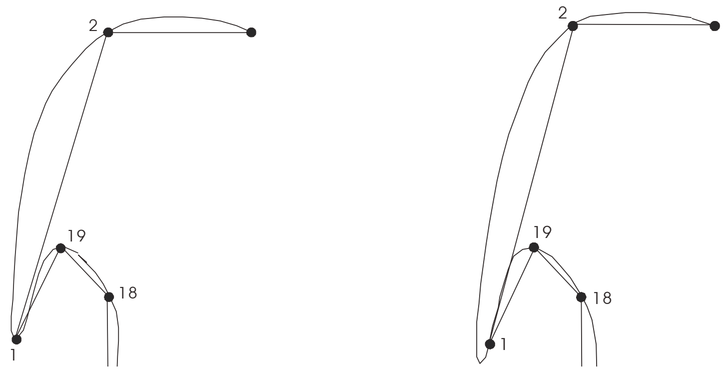


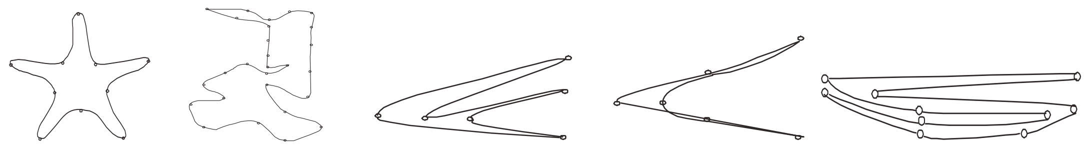
(a) Result of oblique axis parabolic interpolation for curves

(b) The smooth result in the paper of Junkins J.L. & Jancaitis J.R

Figure 9 Local amplification from Figure 8

4.2 Smoothing trial using other typical original data

When the graphic coordinates are relatively dense and change moderately, different methods of smoothing interpolation have similar results. When the graphic coordinates are too dense, any method of smoothing interpolation is no longer needed. Thus, in order to test the adaptability of the algorithm, we need to design a series of scattered points and drastic changes, for graphic testing and comparison. The author has designed some special graph points with which we can test the adaptability of oblique axis parabolic interpolation for curve smoothing (Figure 10).



(a) Test for convex, concave, and symmetric cases

(b) Test for Horizontally, vertically, and collinearly distributed points

(c) Test for sharp round-about curves

(d) Test for sharp turning plus collinearly distributed points

(e) Test for multi zigzag turnings and self-intersection possibilities

Figure 10 Trials on adaptability of oblique axis parabolic interpolation

5 JOINT EXPERIMENTS BY MULTI-UNITS ON CURVE SMOOTHING

In 1977, the author, together with the Cartographic Automation Group of the Institute of Geography of the Chinese Academy of Sciences and Cartographic Automation Teaching Group of Nanjing University, compared the drawing test results of oblique axis parabolic interpolation method with other four main interpolation methods at that time using the typical graphic data designed by the author. Figure 11 shows the results of different interpolation methods [3].

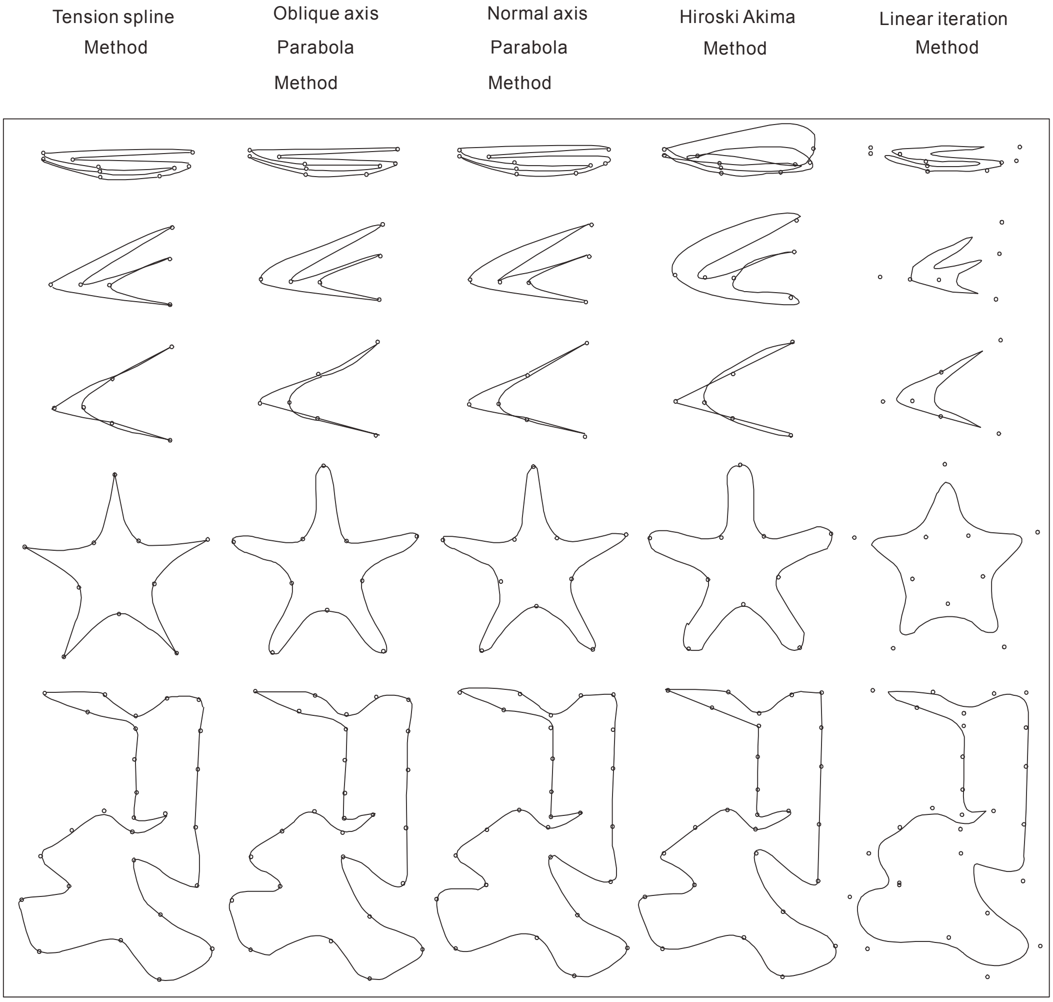


Figure 11 Comparison among five different smoothing methods [3]

6 SCHOLARS' EVALUATIONS ON THE OBLIQUE AXIS PARABOLIC INTERPOLATION METHOD

In the scientific works [3, 4] the authors held the points that:

(1) The oblique axis parabolic interpolation for curve smoothing plays the similar role as the third order polynomial tension Spline interpolation method and the quadratic polynomial weighted average method.

(2) The oblique axis parabolic interpolation method guarantees that the acmes of the parabola always locate at all initial nodes, excluding the endpoints.

(3) As far as the graphic effects are concerned, the oblique axis parabolic interpolation method has more superiorities compared with the previous two smoothing methods (i.e. linear iteration method and Hiroski Akima method [3]). The theory of this method is easier to be understood, because the design of this method was started from the characteristics of curve drawing in cartography, and it ensures the points with local maximal curvatures to locate at all given nodes, excluding only the endpoints.

(4) The interpolated curves between nodes are relatively short.

(5) Curves with zigzag and sharp turning shapes can also have good graphical results, even if the given source points are quite sparse.

(6) The drawback is that the implementation procedure seems to be too complex.

In thesis [5], the authors made a number of comments on oblique axis parabolic interpolation for curve smoothing similar to those in [3, 4] above. In addition, this article proposed and realized the "approximated oblique axis parabolic interpolation for curve smoothing". And it almost has the same result of mapping effect as the oblique axis parabola's introduced in this paper.

According to thesis [6], it is usually recognized that tension Spline interpolation and oblique axis parabolic interpolation are better. Also in the thesis, there was a comparison between these two methods.

Firstly, the authors commented on the principle of oblique axis parabola method similarly as the authors through scientific works [3, 4] did;

Further, the authors of paper [6] commented on the tension spline method as follows: A suitable tension coefficient is needed to be decided. However, the automatic

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determination of this coefficient can hardly fit the practical terrain, while manual selection means too much loss of its efficiency. The fact that all node points of each curve or its local section related must be involved in the calculation of the equation solving has worsened this tendency. This is obviously not economical and time saving. In this thesis, the authors have also optimized the approximation of oblique axis azimuthal angle, interpolation step, as well as conjunctions between straight to curve or curve to straight transitions.

Thesis [7] is about an approach to calculate the derivatives through the oblique axis parabolic method determining the vector direction and its module. That can be called an oblique axis parabola based on Hermite interpolation for curve smoothing.

In many courses of computer cartography, such as in [8-11] etc, there are some similar comments on the oblique axis parabolic interpolation method.

To sum up, scholars' opinions on oblique axis parabolic interpolation for curve smoothing are positive. They have approved its advantages while optimized its over-complex calculation problem from many aspects.

7 TWO ISSUES NEED TO BE FURTHER IMPROVED

7.1 Exploration on simplified exacting solutions of rotation angle of oblique axis parabola

Calculation of the rotation angle of oblique axis parabola can be seen as to solve a canonical cubic algebraic equation first. Since the classical method as Cardan formula to solve the cubic equation is not easy, we used to employ some approximation methods to get the initial value, which can be refined and further approximated through Newton-Raphson iterative tangent method afterwards.

Through a large quantity of experiments, the author has found two simple but applicable methods: Cardan formula based trigonometric solution [12] and Shengjin solution [13] to solve cubic equations. 13 typical cubic equations from the textbooks and difficult curve data have been used to jointly extract solutions employing both of the methods. It ends up with exactly the same results for the two methods.

7.1.1 Simplified solution exaction for cubic equations based on the Cardan formula

Traditional Cardan formula is not applicable to real calculation. The author believes that the formulae in [12, 13] are simpler and more practical.

In the mathematical handbook [12] a Cardan formula based trigonometric solution of cubic equation is as follows:

First, rewrite the equation (1) as

$$ax^3+bx^2+cx+d=0 \tag{3}$$

where

$x=tg\alpha$

Let

$$y=x+\frac{b}{3a}$$

so

$$y^3+3py+2q=0 \tag{4}$$

where

$$\left.\begin{aligned} 3p&=\frac{3ac-b^2}{3a^2} \\ 2q&=\frac{2b^3}{27a^3}-\frac{bc}{3a^2}+\frac{d}{a} \end{aligned}\right\} \tag{5}$$

The number of real roots and their properties of equation (4) are determined by the sign of discriminant $D=q^2+p^3$:

If $D>0$, there are one real root and two imaginary roots;

If $D<0$, there are three different real roots;

If $D=0$ and $p=q=0$, there is one real root (three duplicated zero roots);

When $p^3= - q^2 \neq 0$, there are two real roots (two of the three are duplicated);

Using Cardan formula to solve the cubic equation as (4), there are three different real roots when $D<0$, they are represented in form of complex numbers, this is obviously inconvenient. So the employment of trigonometric auxiliary solution is convenient and practical [12].

In equation (4), taking $R=\pm\sqrt{p}$, the sign of R should be consistent with that of q. With the help of the auxiliary φ, y_1, y_2, and y_3 are determined by the sign of p and the sign of $D=q^2+p^3$ (Table 1). Finally we have that $x=y-b/(3a)$.

When calculating the roots according to table 1, we need to compute φ first, that is, to calculate the inverse function. There is no technical problem in calculating cosφ's inverse function. When the inverse functions of chφ and shφ are to be calculated, some uncommonly seen formulae should be employed:

When calculating the inverse function of chφ, we should use:

$$\left.\begin{aligned} ch\varphi &= \frac{q}{R^3} \\ \varphi = ch^{-1}\varphi &= \ln(ch\varphi+\sqrt{ch^2\varphi-1}) \end{aligned}\right\} \tag{6}$$

When calculating the inverse function of shφ, we should use:

$$\left.\begin{aligned} sh\varphi &= \frac{q}{R^3} \\ \varphi = sh^{-1}\varphi &= \ln(sh\varphi+\sqrt{sh^2\varphi+1}) \end{aligned}\right\} \tag{7}$$

Table 2 Root discriminant of cubic canonical equation (imaginary roots omitted)

p<0		p>0
D<0	D>0	
$\cos\varphi = \frac{q}{R^3}$	$ch\varphi = \frac{q}{R^3}$	$sh\varphi = \frac{q}{R^3}$
$y_1 = -2R\cos(\varphi/3)$ $y_2 = 2R\cos((\pi-\varphi)/3)$ $y_3 = 2R\cos((\pi+\varphi)/3)$	$y_1 = -2Rch(\varphi/3)$	$y_1 = -2Rsh(\varphi/3)$

7.1.2 Shengjin Formula to solve the cubic equation [13]

A young Chinese mathematician, called Fan Shengjin established a general formula which is similar to the solution of quadratic equation to solve the cubic equation, and he also set up a new discriminant criteria. This is not only a new achievement, but also a great contribution to classical mathematics. It has well-manifested the beauty of order, symmetry, harmony and conciseness of mathematics. Here we would rather omit the concrete form of this formula.

7.1.3 Combined calculation test from two methods

Comparing the applications of trigonometric formula based on Cardan formula and Shengjin formula through 13 computational examples, no difference can be found as far as seven decimal places are concerned. This indicates that these two methods are all accurate, convenient and applicable.

In order to avoid the ill conditions for the performance of square, cubic, division and other operations, we should standardize the coordinate values of three neighboring points before solving the cubic equation, to make their absolute values be between 0 and 1. That is, to translate the neighboring points to the location of the second point, then to divide the relative coordinates of points 1 and 3 by the greater side length of the minimal bounded rectangle containing these 3 points.

7.2 Automatic conversion from the tangent to the azimuthal angle

In order to calculate the directional or azimuthal angle satisfying the requirement of oblique axis parabola with the known axial slope of the abscissa x=tgα, the author used to employ trial method [1]. Here the analytical method for automatically determination of azimuthal angle of the abscissa axis is given.

As shown in Figure 12, TT represents the tangent at the point 2. Suppose the slope of the abscissa axis TT of the oblique axis parabola derived from points 1, 2, 3 is: k=tgα. Tangent TT has two possible azimuthal angles. In order to calculate the azimuthal angle satisfying the conditions of the oblique axis parabola, that is, the directional angle which is coincide with the direction of the ordered points (1,2,3,), the author employs the straight line which goes through point 1 and is parallel to TT. It also intersects with the line passing through point 2 and point 3 at point 4, whose coordinates can be computed as:

$$\left.\begin{aligned} x_4 &= \frac{k_{23}x_2 - y_2 - kx_1 + y_1}{k_{23} - k} \\ y_4 &= \frac{k(k_{23}x_2 - y_2) - k_{23}(kx_1 - y_1)}{k_{23} - k} \end{aligned}\right\} \tag{8}$$

In formula (8), k_{23} is the slope coefficient of straight line, passing the points 2 and 3.

Finally, the azimuthal angle of (1, 4) is the angle satisfying the requirement of oblique axis parabola.

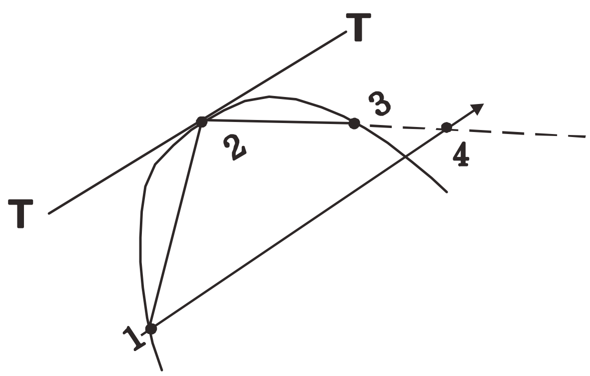

Figure 12 Analytical method of determining the true azimuthal angle of oblique axis parabola

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8 CONCLUSION

Comparing with the author's early paper [1], some information of historical background was added and the important evaluations and approximation approaches toward the oblique axis parabola was summarized. Through analyzing the oblique axis parabola method and the other famous ones, for example, mentioned by [2, 3], and [6], the author and our colleagues have detected that the oblique axis parabola method can guarantee the obvious turning points or the maximal curvature points to locate at the original node points. This method can adapt the case of original sharp turning figures, even for zigzag curves. As a consequence, the obtained interpolated route tends to be the shortest and the possibility of unwanted self-intersection of curves will be minimized. However, for many other methods these features are absent, generally speaking. Finally, the simple and exact methods of solving cubic equations and the new analytical method of computing the true azimuthal angle of oblique axis parabola, together make the oblique axis parabola interpolation method be more sophisticated.

AUTHOR IN BRIEF

WU Hehai, Professor of Wuhan University, Ph.D. supervisor, with main research directions: cartography, GIS and fundamental models and algorithms of automatic generalization.

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